Econ 110/PoliSci 135
Section 9 Notes

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1 Repeated Games

There are two types of repeated games: infinitely repeated games, and finite repeated games.

1.1 Infinitely Repeated Games

The static game that we play over and over again is known as the **stage game**. The stage game is sometimes also called the **one-shot game** or the **constituent game**. We're going to have to be very careful know how we specify strategies. Remember, a strategy is a *complete* plan for play, under all possible circumstances. Since there are now infinitely-many repetitions, there are infinitely-many possible histories and hence infinitely-many information sets at which we have to specify an action.

One other conceptually-difficult task we'll have to perform is to think about one player deviating, holding the other player's strategy constant. This is the whole key to equilibrium in game theory, so make sure you're comfortable with it. We only consider *unilateral* deviations. Anything more than that would make our heads explode.

1.1.1 "Grim Trigger" Strategies

An important class of equilibria in infinitely-repeated games is that which has the players using "trigger strategies" to guarantee a certain outcome of the stage game. Player i's strategy is given by the following:

- If the other player has always played the action she's "supposed to be playing" (i.e. cooperated), then play what I'm "supposed to play" (i.e. cooperate as well).
- If the other player has ever deviated, play my minmax strategy forever.

To determine if a cell in the stage game can be the "cooperate" part of a grim trigger strategy, we need to look at four variables:

1. V_c = the stage game payoff from the cooperative action
2. V_d = the stage game payoff from the best other action, given that the opponent is still cooperating (remember, we only think about unilateral deviations)
3. V_p = the stage game payoff from being "punished", i.e. a player's minmax payoff.
4. δ = the discount factor (the higher this is, the more patient the player is)

A grim trigger strategy can enforce this cooperative strategy if neither player wants to unilaterally deviate. Let's write the inequality that expresses greater utility for cooperating than deviating. If a player cooperates forever, then that player gets V_c today, and V_c tomorrow (discount by δ) and V_c the next day, and so on forever.

The best deviation a player could make today would give V_d. But since that player deviated, the other player will punish forever. This will give the deviating player the minmax payoff in every subsequent stage game.

Thus, a player will cooperate forever if and only if:

$$\begin{aligned} U(cooperating\ forever) &\geq U(deviating\ once\ and\ being\ punished\ ever\ after) \\ V_c + \delta \cdot V_c + \delta^2 \cdot V_c + \cdots &\geq V_d + \delta V_p + \delta^2 V_p + \cdots \\ V_c \cdot \frac{1}{1-\delta} &\geq V_d + V_p \cdot \frac{\delta}{1-\delta} \end{aligned}$$

We can rewrite this final inequality as

$$\frac{\delta}{1-\delta}[V_c - V_p] \geq V_d - V_c.$$

The left-hand side is the future benefit of staying in the cooperative regime over the punishment regime, discounted over time. The right-hand side is the present benefit of defecting over cooperating. If the future benefits are larger than the immediate gains, a player won't defect.

1.2 An example

Return to the matrix from section 2.1. Let's construct a grim trigger strategy for both players and check for which discount factor cooperation can be sustained.

An outcome is **Individually Rational** if every player's payoff is *strictly larger* than its minmax payoff. So our matrix has four Individually Rational Outcomes: (5,5), (6,8), (7,7), and (8,8).

Let's construct an equilibrium where both players are playing Grim Trigger Strategies that supports cooperating on the (7,7) outcome.

Grim Trigger Strategies:

- For Player 1: Cooperate in Round 1. As long as P2 has cooperated in the last round, keep playing Cooperate in the current round. Otherwise minmax Player 2 forever.
- For Player 2: Cooperate in Round 1. As long as P1 has cooperated in the last round, keep playing Cooperate in the current round. Otherwise minmax Player 1 forever.

Is this strategy profile Nash? First, notice that if both players are playing (B,H), then Player 2 is the only player who can profitably deviate from this stage game Nash. To see for what δ Player 2 will cooperate forever, we have to set up the following inequality:

$$EU_2(Cooperate) \geq EU_2(Deviate)$$
$$\tfrac{7}{1-\delta} \geq 8 + \tfrac{3\delta}{1-\delta}$$
$$\delta \geq 1/5$$

So Players 1 and 2 can Cooperate on the outcome of (7,7) as long as $\delta \geq 1/5$.

Folk Theorem: We can construct a Nash Equilibrium such that players will play an Individually Rational strategy given that δ is large enough.

2 Finitely-Repeated Games

Thus far all of our analysis in repeated games has been with regards to finding *Nash Equilibria* and not *subgame perfect equilibria*. Now, we will continue to solve for Nash Equilibria in finitely-repeated games before we distinguish Nash from subgame perfect equilibria.

In another of the counter-intuitive parts of Game Theory, the analysis of infinitely-repeated games was actually easier than the analysis of finitely-repeated ones. Before, the strength of a threat didn't depend on when the players were thinking about deviating – an infinite punishment is infinite in any round. Now, the closer we get to the end of the game, the weaker a threat becomes.

In the final round, there's no ability to make a threat at all. Since all a threat does is promise bad payoffs in the future (and there's no more future in the last round), players will rationally play their best-response to what they think their opponent is doing. So that means, in the final stage, players will play the stage Nash Equilibrium.

For this course, we'll look at a small class of equilibria. The equilibrium is composed of two phases. In the late part of the game, we'll have the "non-cooperative" regime – players play the stage game Nash Equilibrium strategies. In the early part of the game, we'll have the "cooperative" regime – players agree on what cell to play, and punish deviations by minmaxing.

2.1 Example

Suppose two firms are competing in differentiated Bertrand competition, say Intel and AMD selling their latest processors. Each firm can charge a low price, charge a high price, or dump their product at less than cost. Due to the limited commercial lifetime of a computer processor, this market interaction will be repeated every month but only for 24 months. The discount rate is 0.9.

	D	**L**	**H**
D	-2,-2	-2,0	-2,0
L	0,-2	2,2	8,1
H	0,-2	1,8	4,4

Clearly the two firms would prefer to sustain the collusive outcome of (H,H) for as long as possible. Using a minmax-threat strategy, how long can the "cooperative" regime be?

The first step is of course to find the minmax strategies and payoffs. While we're at it, we can find the Nash strategies. The underlining method quickly reveals the minimax strategies to be D and D, for payoffs of 0 and 0. The stage game Nash equilibrium is (L,L), for payoffs of 2,2.

Let's start specifying strategies, working from the end back. Each player in round 24 will play L – the Nash strategy. Can we start the cooperative regime in round 23? If we could, players would have to want to play H instead of deviating in this round. Let's compare payoffs:

If a player "cooperates" by playing H, he gets 4 in this round, plus 2 from the NE in the next round for a total of $4 + 2\delta = 5.8$.

If a player "deviates", his best deviation is to play L to get 8 in this round. Then next round he'd be minmaxed, and recieve 0. The total payoff is $8 + 0\delta = 8$, which is greater than the payoff to cooperating.

If the 23rd round is too late for the cooperative regime, maybe the 22nd round will be ok. Comparing payoffs here, we get $4 + 2\delta + 2\delta^2 = 7.42$ for cooperating and $8 + 0\delta + 0\delta^2 = 8$ for deviating – a player would still deviate here.

How about round 21? Cooperating leads to $4 + 2\delta + 2\delta^2 + 2\delta^3 = 8.149$. Deviating still leads to 8. Finally, we have cooperation.

Thus players' strategies are:

In rounds 1-21, play H if opponent has always played H. Play D otherwise.

In rounds 22-24, play L if opponent has played H in 1-21 and L 22-now. Play D otherwise.

3 Subgame Perfection in Repeated Games

Now let's take some time to think about what happens when threats have to be credible. Just like in one-stage games, we formalize this in the notion of subgame perfect equilibria. Remember the definition:

Subgame Perfect Equilbrium A Nash Equilibrium is subgame-perfect if it forms a Nash equilibrium in *every* subgame.

The same definition works in this more complicated setting, but obviously it will be more complicated to verify when we have a SPE. Just be very careful to consider *all* subgames (either one at a time, or by somehow dividing them into categories), and you'll be fine.

The big problem with the grim trigger strategies we've been using before is that they aren't subgame perfect. The way we found a player's minmax strategy said nothing about his own payoffs, which means that it is likely that she has an incentive to deviate in the subgames where she's supposed to punish the other player.

The good news is that there's an equally-simple strategy that we can use instead that *is* subgame-perfect: the Nash Threat trigger strategy. There are two main differences from the "grim triggers". First, the punishment is now always playing a Nash equilbrium. Second, we change the trigger subtly to take care of the subgame that starts when the punishment is first triggered:

Nash Trigger Threat Strategy: Choose a set of stage-game actions, and designate them as "cooperation". A player's Nash Threat strategy is to always play the cooperative action so long as both players have always cooperated. If either player has ever deviated, play the stage game Nash forever.

Because this is a weaker threat than the grim trigger strategy, it will be able to support a smaller set of stage outcomes. On the other hand, because the strategies are subgame perfect, the claim to punish forever is credible.