August 2015

MATH CAMP

Problem Set 3

1. Suppose the probability of observation O_1 given hypothesis H_1 is $P(O_1 | H_1) = 1/3$, the probability of observation O_2 given hypothesis H_1 is $P(O_2 | H_1) = 2/3$, the probability of observation O_1 given hypothesis H_2 is $P(O_1 | H_2) = 2/3$, and the probability of observation O_2 given hypothesis H_2 is $P(O_2 | H_2) = 1/3$. The prior probabilities of hypotheses H_1 and H_2 are $P(H_1) = 1/2$ and $P(H_2) = 1/2$. Research then yields observation O_1. Given this observation, what is the probability of H_1, i.e., what is $P(H_1 | O_1)$?

2. A certain component in a rocket engine fails 5% of the time when the engine is fired. To achieve greater reliability in the engine working, this component is duplicated n times. The engine then fails only if all of these n components fail. Assume the component failures are independent of each other. What is the smallest value of n that can be used to guarantee that the engine works 99% of the time?

3. S&B A5.2, p. 897

4. S&B A5.3, p. 897

5. The random variable *x* can take on the values -2, 0, and 2 with probabilities 1/6, 1/6, and 4/6 respectively. What is the expected value of *x*? What is the variance of *x* (write an expression for calculating the variance but you do not need to do the arithmetic of actually calculating it)?

6. G 3-6, p. 136.

7. The random variable *z* is distributed according to the density function:

$$f(z) = \begin{cases} 0 & \text{if } z < -1 \\ h(1 - z^2) & \text{if } -1 \leq z \leq 1 \\ 0 & \text{if } 1 < z \end{cases}$$

where *h* is a parameter to be determined.

(a) What is the value of *h*?

(b) Find the cumulative distribution function $F(z)$ for any *z* between -1 and 1.

(c) What is the probability that *z* is between 0 and 1/2?

8. Spend no more than an hour trying to show that the variance of a normally distributed random variable with mean μ actually is σ^2. That is, show that

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \sigma^2$$

where, recall, the expression on the left is the density of a random variable distributed normally with mean μ and variance σ^2.

Hint: Observe that

$$\frac{d((x-\mu)e^{-(x-\mu)^2/(2\sigma^2)})}{dx} = e^{-(x-\mu)^2/(2\sigma^2)} - \frac{(x-\mu)^2}{\sigma^2}e^{-(x-\mu)^2/(2\sigma^2)}.$$

Integrating both sides of this equation then yields:

$$\int_{-\infty}^{\infty} d((x-\mu)e^{-(x-\mu)^2/(2\sigma^2)}) = \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}dx - \int_{-\infty}^{\infty}\frac{(x-\mu)^2}{\sigma^2}e^{-(x-\mu)^2/(2\sigma^2)}dx$$

$$(x-\mu)e^{-(x-\mu)^2/(2\sigma^2)}\Big|_{-\infty}^{\infty} = \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}dx - \int_{-\infty}^{\infty}\frac{(x-\mu)^2}{\sigma^2}e^{-(x-\mu)^2/(2\sigma^2)}dx$$

$$\frac{1}{\sigma^2}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-(x-\mu)^2/(2\sigma^2)}dx = \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}dx - (x-\mu)e^{-(x-\mu)^2/(2\sigma^2)}\Big|_{-\infty}^{\infty}$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-(x-\mu)^2/(2\sigma^2)}dx = \frac{\sigma}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}dx - (x-\mu)e^{-(x-\mu)^2/(2\sigma^2)}\Big|_{-\infty}^{\infty}\right]$$

Can you now show that the expression on the right of the equals sign reduces to σ^2?